UNIT-II

VECTOR CALCULUS
PART-A

1. Is the vector $xi + 2yj + 3zk$, **Irrotational?** **(AU-2009)**

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & 2y & 3z \end{vmatrix} = i\ (0-0) - j\ (0) + k\ (0-0) = 0$$

$\therefore$ $\vec{F}$ is irrotational.

2. Find the divcurl $\vec{F} = x^2 y\vec{i} + xz\vec{j} + 2yz\vec{k}$ **(AU-2010)**

$$\text{Curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 y & xz & 2yz \end{vmatrix} = \vec{i}\left[\frac{\partial}{\partial y}(2yz) - \frac{\partial}{\partial z}(xz)\right] - \vec{j}\left[\frac{\partial}{\partial x}(2yz) - \frac{\partial}{\partial z}(x^2 y)\right] + \vec{k}\left[\frac{\partial}{\partial x}(xz) - \frac{\partial}{\partial y}(x^2 y)\right]$$

$$= \vec{i}(2z - x) - \vec{j}(0) + \vec{k}(z - x^2)$$

$$\text{Divcurl } \vec{F} = \nabla . \text{curl } \vec{F}$$

$$= \nabla.\left(i(2z-x) - j(0) + k(z - x^2)\right)$$

$$= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\left(i(2z - x) - j(0) + k(z - x^2)\right)$$

$$= \left(\frac{\partial}{\partial x}(2z - x) + \frac{\partial}{\partial z}(z - x^2)\right)$$

$$= -1 + 1 = 0$$

3. If ∇F **=yz** i **+xz** j **+xy** k **then find F** **(AU-2010)**

$$\nabla F = \text{yz } i + \text{xz } j + \text{xy } k$$

$$i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z} = \text{yz } i + \text{xz } j + \text{xy } k$$

Equating the coefficient of i, j, k

$$\frac{\partial f}{\partial x} = yz, \quad \frac{\partial f}{\partial y} = xz, \quad \frac{\partial f}{\partial z} = xy$$

$$\int \partial f = \int yz\partial x \qquad \int \partial f = \int xz\partial y \qquad \int \partial f = \int xy\partial z$$

f_1=xyz+f(y,z) f_2=xyz+f(x,z) f_1=xyz+f(x,y)

F=xyz+c

4. **Find the unit normal to the surface** $x^2 + y^2 - 2z + 3 = 0$ **at (1, 2,-1)**

Given $\Phi = x^2 + y^2 - 2z + 3 = 0$

$$\nabla\phi = \left(\frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j + \frac{\partial\phi}{\partial z} k\right)(x^2 + y^2 - 2z + 3)$$

$$= 2xi + 2yj - 2k$$

$$(\nabla\phi)_{(1,2,-1)} = (2i + 4j - 2k), \quad |\nabla\phi| = 2\sqrt{6}$$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2i + 4j - 2k}{2\sqrt{6}}$$

5. In what direction from (3,1,-2) is the directional derivative of $\phi = x^2 y^2 z^4$ maximum? Find Also the magnitude of this maximum. **(AU-2015)**

$\nabla\phi = 2xy^2 z^4 \vec{i} + 2x^2 yz^4 \vec{j} + 4x^2 y^2 z^3 \vec{k}$

At(3,1,-2), $\nabla\phi = 96\left(\vec{i} + 3\vec{j} - 3\vec{k}\right)$

Direction of Maximum = $\nabla\phi = 96\left(\vec{i} + 3\vec{j} - 3\vec{k}\right)$

Magnitude= $|\nabla\phi| = 96\sqrt{1+9+9} = 96\sqrt{19}$.

6. Prove that $\vec{F} = yz\,\vec{i} + zx\,\vec{j} + xy\,\vec{k}$ is irrotational. **(AU-2012)**

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial y} \\ yz & zx & xy \end{vmatrix}$$

$\vec{i}\,(x - x) - \vec{j}(y - y) + \vec{k}\,(z - z) = 0$

∴ $\vec{F}$ is irrotational.

7. Find λ so that $\vec{F} = (3x - 2y + z)\vec{i} + (4x + \lambda y - z)\vec{j} + (x - y + 2k)\vec{k}$ is solenoidal (AU-2015)-2

Given $\vec{F}$ is solenoidal then $\nabla \cdot \vec{F} = 0$

$$\nabla\cdot\vec{F} = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial y}\vec{k}\right)\cdot\left((3x - 2y + z)\vec{i} + (4x + \lambda y - z)\vec{j} + (x - y + 2k)\vec{k}\right)$$

$3 + \lambda + 2 = 0$

$\lambda = -5$

8. If A and B are irrotational , prove that $A \times B$ is solenoidal. **(AU-2013)**

If A and B are irrotational.

$\nabla\times A = 0, \nabla\times B = 0$

We know that $\nabla.(A\times B) = (\nabla\times A).B - (\nabla\times B).A = 0 - 0 = 0$

$\therefore A\times B$ is solenoidal.

9. Define solenoidal vector function. If $\overline{V} = (x + 3y)i + (y - 2z)j + (x + 2\lambda z)k$ is solenoidal, then find the value of λ **(AU-2013)**

Given that $\overline{V} = (x + 3y)i + (y - 2z)j + (x + 2\lambda z)k$

$\nabla.\overline{V} = 0$ if $\overline{V}$ is solenoidal

$$\nabla.\overline{V} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\cdot(x + 3y)i + (y - 2z)j + (x + 2\lambda z)k = 0$$

$$= \left(\frac{\partial}{\partial x}(x + 3y) + \frac{\partial}{\partial y}(y - 2z) + \frac{\partial}{\partial z}(x + 2\lambda z)\right) = 0$$

$= 1 + 1 + 2\lambda = 0$

$\lambda = -1$

10. Find the value of the constant a, b, c so that the vector

$F = (x + 2y + az)i + (bx - 3y - z)j + (4x + cy + 2z)k$ **is irrotational** **(AU-2010)**

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix}$$

$$= i\left[\frac{\partial}{\partial x}(4x+cy+2z) - \frac{\partial}{\partial z}(bx-3y-z)\right] - j\left[\frac{\partial}{\partial x}(4x+cy+2z) - \frac{\partial}{\partial z}(x+2y+az)\right] + k\left[\frac{\partial}{\partial x}(bx-3y-z) - \frac{\partial}{\partial y}(x+2y+az)\right]$$

$$= i(c+1) - j(4-a) + k(b-2)$$

Given F is irrotational, $\nabla\times F = 0$.

$$i(c+1) - j(4-a) + k(b-2) = 0$$

Each component should be zero.

C+1=0, a-4 =0, b-2 = 0

C=-1, a=4, b=2.

11. Prove that $\nabla . r^n = nr^{n-2} . r$ **(AU-2011)**

Let $r = xi + yj + zk$, $|r| = \sqrt{x^2 + y^2 + z^2}$,

$$\nabla . r^n = i\frac{\partial}{\partial x}(r^n) + j\frac{\partial}{\partial y}(r^n) + k\frac{\partial}{\partial z}(r^n)$$

$$= i\left[nr^{n-1}\frac{\partial r}{\partial x}\right] + j\left[nr^{n-1}\frac{\partial r}{\partial y}\right] + k\left[nr^{n-1}\frac{\partial r}{\partial z}\right]$$

$$= nr^{n-1}\left[i\frac{\partial r}{\partial x} + j\frac{\partial r}{\partial y} + k\frac{\partial r}{\partial z}\right]$$

$$= nr^{n-1}\left[i\frac{x}{r} + j\frac{y}{r} + k\frac{z}{r}\right]$$

$$= nr^{n-1}\frac{1}{r}\left[xi + yj + zk\right]$$

$$= nr^{n-2} r$$

12. Find the directional derivative $\phi = x^2 + y^2 + z^2$ in the direction of the vector **(AU-2014)**

$\vec{F} = \vec{i} + 2\vec{j} + 2\vec{k}$ at (1,1,1)

Unit normal vector $\hat{n}$ in the direction of $\vec{i} + 2\vec{j} + 2\vec{k}$ **is** $\left(\frac{\vec{i} + 2\vec{j} + 2\vec{k}}{3}\right)$

$$\text{Directional derivative} = \nabla\phi \cdot \hat{n}$$

$$\nabla\phi = \vec{i}\frac{\partial}{\partial x}(x^2+y^2+z^2) + \vec{j}\frac{\partial}{\partial y}(x^2+y^2+z^2) + \vec{k}\frac{\partial}{\partial y}(x^2+y^2+z^2)$$

$$grad\ \phi = 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$$

$$\nabla\phi_{at(1,1,1)} = 2\vec{i} + 2\vec{j} + 2\vec{k}$$

$$\text{Directional derivative} = \nabla\phi\cdot\hat{n} = \left(2\vec{i} + 2\vec{j} + 2\vec{k}\right)\left(\frac{\vec{i} + 2\vec{j} + 2\vec{k}}{3}\right) = \frac{10}{3}$$

13. Find the unit normal vector to the surface $x^2+y^2=z$ at (1,-2,5) **(AU-2014)**

$\square = \mathbf{x^2+y^2-z}$

$$\nabla\phi = \left(\frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k\right)x^2+y^2-z$$

$$= 2xi + 2yj - k$$

$$(\nabla\phi)_{(1,-2,5)} = (2i - 4j - k),\ |\nabla\phi| = \sqrt{21}$$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2i-4j-k}{\sqrt{21}}$$

(AU-2009)

14. Show that $F = (x^2 i + y^2 j + z^2)k$ is a conservative vector field.

If $\hat{F}$ is conservative then $\nabla \times \hat{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & y^2 & z^2 \end{vmatrix} = 0i + 0j + 0k = 0$

Therefore $\hat{F}$ is a conservative vector field.

(AU-2014)

15. Prove that curl(grad ϕ)=0

$$\text{curl(grad }\phi) = \text{curl}(\nabla\phi) = \nabla \times \nabla\phi = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) \times \left(\frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k\right)$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix} = \vec{i}\left[\frac{\partial^2\phi}{\partial y\partial z} - \frac{\partial^2\phi}{\partial y\partial z}\right] - \vec{j}\left[\frac{\partial^2\phi}{\partial x\partial z} - \frac{\partial^2\phi}{\partial x\partial z}\right] + \vec{k}\left[\frac{\partial^2\phi}{\partial y\partial x} - \frac{\partial^2\phi}{\partial y\partial x}\right] = 0$$

(AU-2014)

16. Find $Curl\,\vec{F}$ if $\vec{F} = xy\vec{i} + yz\vec{j} + zx\vec{k}$ (AU-2010)

$$Curl\ \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & yz & zx \end{vmatrix} = \vec{i}(-y) - \vec{z}j - \vec{x}k$$

17. If $F = x^2 i + xyj$ evaluate $\int_c F.dr$ from (0,0) to (1,1) along the line y = x.

Given $F = x^2 i + xyj$

Along the line y = x, dy = dx

$$\therefore F = x^2 i + x.xj, \quad dr = dxi + dyj = dxi + dxj$$

$$F.dr = (x^2 i + x^2 j).(dxi + dyj)$$

$$= x^2dx + x^2dx = 2x^2dx$$

$$\int_c F.dr = \int_0^1 2x^2\,dx = \frac{2}{3}$$

18. If $F = 5xyi + 2yj$, evaluate $\int_c F.dr$ Where C is the part of the curve $y = x^2$ between $x = 1$ and $x = 2$.

(AU-2012)

$F.dr = (5xyi + 2yj).(dxi + dyj +$ = 5xydx+2ydy

The curve C: $y = x^2$ *dzk*)

$dy=2xdx$

x varies from 1 to 2

$$\int_c F.dr = \int_1^2 5x(x^2)dx + 2x^2 \, 2xdx = 5\left[\frac{x^4}{4} + \frac{4x^4}{4}\right]_1^2$$

$$= 36 - \frac{9}{4} = \frac{135}{4}$$

19. Find $\iint_s r.ds$ where S the surface of the tetrahedron whose vertices are is

(0, 0, 0), (1, 0, 0), (0,1, 0), (0, 0,1). **(AU-2010)**

By Gauss divergence theorem

$$\iint_S r.ds = \iiint_V \nabla.r dv$$

$$= \iiint_V \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right).(xi + yj + zk)dv$$

$$= \iiint_V (1+1+1)dv$$

$$= \iiint_V 3dv$$

$$= 3\int_0^1\int_0^1\int_0^1 dxdydz = 3.$$

20. If S is any closed surface enclosing a volume V and $F = ax\,i + by\,j + cz\,k$, prove that $\iint_S F.\hat{n}ds$ =(a+b+c)V. **(AU-2011)**

Gauss Divergence theorem is

$$\iint_S F.\hat{n}.ds = \iiint_V \nabla.F dV$$

$$= \iiint_V \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right).(axi + byj + czk)dv$$

$$= \iiint_V (a+b+c)dv$$

= (a+b+c) V

21. State Green's theorem in a plane. **(AU-2010)** If M(x,y) and N(x,y) and its partial derivatives are continuous and one valued functions in the region R of the xy plane bounded by a simple closed curve C, then

$$\int_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dxdy$$

Where C is the curve prescribed in the positive direction.

22. Using Green's theorem, Prove that the area enclosed by a simple closed curve C is $\frac{1}{2}\int(xdy - ydx)$. **(AU-2011)**

By Green's theorem

$$\int_C Mdx + Ndy = \iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dxdy$$

Let M = -y N=x

$$\int_C -ydx + xdy = \iint_S (1+1)dxdy$$

$$= 2\iint_S dxdy$$

= 2(area enclosed by C)

Therefore Area enclosed by C = $\frac{1}{2}\int(xdy - ydx)$

23. State Gauss Divergence theorem. **(AU-2012)**

If V is the volume bounded by a closed surface S and if a vector function $\vec{F}$ is continuous and has continuous partial derivatives in V and on S then

$$\iint_s \vec{F}\cdot\hat{n}\,ds = \iiint_v \nabla\cdot\vec{F}\,dv$$

24. State Stoke's theorem. **(AU-2015) (2)**

The surface integral of the normal component of the curl of a vector function F over an Open surface S is equal to the line integral of the tangential component of F around the Closed curve C bounding S. $\int_C F.dr = \iint_S \nabla\times F.\hat{n}\,ds$

PART-B

1. a. If $r = xi + yj + zk$ and $r = |r|$. Prove that $div(r^n r) = (n+3)r^n$ and $curl(r^n r) = 0$. **(AU-2011) (8)**

   b. If $r = xi + yj + zk$, then prove that $div\ grad\ (r^n) = n(n+1)r^{n-2}$ .Hence deduce that $div grad\left(\frac{1}{r}\right) = 0$ **(AU-2015)-2(8)**

2. a. Find the directional derivative of ϕ=3x^2+2y-3z at (1,1,1) in the direction of $2i + 2j - k$ **(AU-2012) (8)**

   b. Find the angle between the surfaces x^2+y^2+z^2=9 and z=x^2+y^2-3 at the point (2,-1,2). **(AU-2012)(8)**

3. a. Find the angle between the normal's to the surfaces x^2=yz at the points (1,1,1) and(2,4,1) **(AU-2014)(8)**

   b. Find the directional derivative of ϕ=x^{2}yz+4xz^2 at P(1,-2,-1) that is maximum and also in the direction of PQ where Q is (3,-3,-2) **(AU-2010 (8)**

4. a. Evaluate $\int_C \phi d\vec{V}$ where C is the curve x=t, y=t^2 , z=1-t and $\phi = x^2 y(1+z)$ from t=0 to t=1 **(AU-2011)(8)**

   b. If $\nabla\phi = (x + 2y + 4z)\vec{i} + (2x - 3y - z)\vec{j} + (4x - y + 2z)\vec{k}$, find the Scalar point functionϕ . **(AU-2011)(8)**

5. a. Find the value of n so that the vector $r^n \boldsymbol{r}$ is both solenoidal and irrotational **(AU-2015)-2(8)**

` b. Prove that $\vec{F} = (x^2 - y^2 + x)\vec{i} - (2xy + y)\vec{j}$ is irrotational and hence find its scalar potential. **(AU-2014)(8)**

6. a. Prove that $\vec{F} = (6xy + z^3)\vec{i} + (3x^2 - z)\vec{j} + (3xz^2 - y)\vec{k}$ is irrotational .Hence find its scalar potential φ **(AU-2015)(8)**

   b. Prove that $\vec{F} = (y^2 + 2xz^2)\vec{i} + (2xy - z)\vec{j} + (2x^2 z - y + 2z)\vec{k}$ is irrotational and hence find its scalar potential. **(AU-2014)(8)**

7. a. Find the work done the force $F = 3xyi - y^3 j$ moves a particle along the Curve C: y=2x^{2}from (0, 0) to (1, 2) in the xy-plane. **(AU-2011)(8)**

   b. Evaluate $\int_c f..dr$ where $f = (2xy + z^3)i + x^2 j + 3xz^2 k$ along the straight line Joining (1, - 2,1) and (3, 2, 4) **(AU-2012)(8)**

8. a. Show that $\vec{F} = (2xy + z^3)\vec{i} + x^2\vec{j} + 3xz^2\vec{k}$ is a conservative force field. Find the Scalar potential and the work done by $\vec{F}$ in moving an object in this field

from (1,-2, 1) to (3, 1,4) **(AU-2009)(8)**

b. Find the directional derivative of xy^2+yz^3 at (2,-1,1) in the direction of the normal to the surface $x\log z-y^2+4=0$ at (-1,2,1) **(AU-2009)(8)**

9. a .Evaluate $\iint_S f\cdot\hat{n}\,ds$ Where $f=(x^2+y^2)i-2xj+2yzk$ and S is the surface of the $2x+y+2z=6$ in the first octant. **(AU-2010)(8)**

b. Using Green's theorem in the plane evaluate $\int_C (3x^2-8y^2)dx+(4y-6xy)dy$ Where C is the boundary of the region enclosed by $y=\sqrt{x}$ and $y=x^2$. **(AU-2009)(8)**

10. a. Using Green's theorem ,evaluate $\int_C (y-\sin x)dx+\cos x\,dy$ where C is the triangle formed by y=0, , $x=\frac{\pi}{2}$, $y=\frac{2x}{\pi}$ **(AU-2015)(8)**

b. Apply Green's theorem in the plane to evaluate $\int_C (3x^2-8y^2)dx+(4y-6xy)dy$ Where C is the boundary of the region defined by by $x=0, y=0$ and $x+y=1$. **(AU-2014)-2(8)**

11.a. Verify Green's theorem in a plane for $\int (xy+y^2)dx+x^2dy$ where C is the boundary of the common area between $y=x^2$ and y=x in the xoy plane **(AU-2014)(8)**

b.Using Green's theorem ,evaluate $\int_C (x^2-2xy)dx+(x^2y+3)dy$,where C is the region bounded by the curves $y^2=8x$ and x=2 **(AU-2015)(8)**

12.a. Verify Gauss Divergence theorem for $F=x^2i+y^2j+z^2k$ Where S is the surface of the Cuboid formed by the planes $x=0, x=a, y=0, y=b, z=0, z=c$. **(AU-2014)(8)**

b. Verify Gauss's divergence theorem for the function $F=y\,i+x\,j+z^2k$ Over the cylindrical region bounded by $x^2+y^2=9$, $z=0$ and $z=2$. **(AU-2012)(8)**

13. Verify Gauss's divergence theorem for $F=4xz\,i-y^2j+yz\,k$ and C is it's boundary over the cube $x=0, x=1, y=0, y=1, z=0, z=1$. **(AU-2015)-3(16)**

14. Verify Gauss Divergence theorem for $F=x^2i+y^2j+z^2k$ taken over the cube bounded by the Planes x=0,y=0,z=0,x=1,y=1 and z=1 **(AU-2015)(16)**

15.a. Verify stoke's theorem for $F=(x^2-y^2)\,i+2xy\,j$ in the rectangular region in the xy plane bounded by the lines $x=0, x=a, y=0, y=b$. **(AU-2015)-3(8)**

b. Verify Stoke's theorem for the vector field $\vec{F}=(2x-y)\vec{i}-yz^2\vec{j}-y^2z\vec{k}$ where S is the surface of upper hemisphere $x^2+y^2+z^2=1$ and C is its boundary in xy-plane. **(AU-2014)(8)**

UNIT-V
LAPLACE TRANSFORMS
PART-A

1. **State the sufficient condition for existence of the Laplace transform of f(t)** (i)f(t)should be continuous or piecewise continuous in the given closed interval [a,b] where a>0 **(AU-2015)**
(ii) f(t) should be of exponential order.

2. **Find the Laplace transform of** $f(t) = t\cosh t$ **(AU-2014)**

$$L(t\cosh t) = -\frac{d}{ds}\left(L(\cosh t)\right) = -\frac{d}{ds}\left[\frac{s}{s^2-1}\right]$$

$$= -\left[\frac{(s^2-1)(1)-s(2s)}{(s^2-1)^2}\right] = -\left[\frac{-1-s^2}{(s^2-1)^2}\right] = \frac{s^2+1}{(s^2-1)^2}$$

(AU-2013)

3. **Find the Laplace transform of** $\frac{t}{e^t}$

$$L\left(\frac{t}{e^t}\right) = L\left(te^{-t}\right) = L(t)_{s\to s+1} = \left(\frac{1}{s^2}\right)_{s\to s+1} = \frac{1}{(s+1)^2}$$

(AU-2012)

4. **State and prove change of scale property in Laplace transform.**

If L(f(t)) = F(s), then L(f(at)) = $\frac{1}{a}F\left(\frac{s}{a}\right)$.

L(f(at)) = $\int_0^\infty e^{-st} f(at)dt$

at=u t=0 u=0

adt=du t=∞ u=∞

L(f(at)) = $\int_0^\infty e^{-\frac{su}{a}} f(u)\frac{du}{a}$

$= \frac{1}{a}\int_0^\infty e^{-\frac{su}{a}} f(u)du$

L(f(at)) $= \frac{1}{a}F\left(\frac{s}{a}\right)$

(AU-2012)

5. **State the first shifting theorem on Laplace transforms.**

If L(f(t))=F(s) then L[e^{at}f(t)]=F[s-a] and

If L(f(t))=F(s) then L[e^{-at}f(t)]=F[s+a]

(AU-2012)

6. **Find the Laplace transform of** $\frac{e^{-2t}}{\sqrt{t}}$

$$L\left[t^{-1/2}e^{-2t}\right] = L\left[t^{-1/2}\right]_{s\to s+2}$$

$$= \left[\frac{\sqrt{\pi}}{\sqrt{s}}\right]_{s\to s+2}$$

$$= \left[\frac{\sqrt{\pi}}{\sqrt{s+2}}\right]$$

(AU-2012)

7. **Find the Laplace transform of** $\sqrt{t}e^{3t}$

$$L\left[t^{1/2}e^{3t}\right] = L\left[t^{1/2}\right]_{s\to s-3}$$

$$= \left[\frac{\sqrt{\pi}}{2s^{3/2}}\right]_{s\to s-3}$$

$$= \frac{\sqrt{\pi}}{2(s-3)^{3/2}}$$

8.Find $L\left[\cos^2 3t\right]$ **(AU-2011)**

$$L[\cos^2 3t] = L\left[\frac{1+\cos 6t}{2}\right]$$

$$= \frac{1}{2}L[1+\cos 6t]$$

$$= \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+36}\right]$$

9. Find $L\left[\left(t^3 + 3e^{2t} - 5\sin 3t\right)e^{-t}\right]$ **(AU-2011)**

$$L\left[\left(t^3 + 3e^{2t} - 5\sin 3t\right)e^{-t}\right] = L\left[\left(t^3e^{-t} + 3e^{2t}e^{-t} - 5\sin 3te^{-t}\right)\right]$$

$$= L\left[\left(t^3e^{-t} + 3e^{t} - 5\sin 3te^{-t}\right)\right]$$

$$= L\left[t^3\right]_{s\to s+1} + 3L(1)_{s\to s+1} - 5L[\sin 3t]_{s\to s+1}$$

$$= \left[\frac{6}{s^4} + \frac{3}{s} - \frac{3}{s^2+9}\right]_{s\to s+1}$$

$$= \left[\frac{6}{(s+1)^4} + \frac{3}{s+1} - \frac{3}{(s+1)^2+9}\right]_{s\to s+1}$$

10. Find $L\left[\frac{\sin t}{t}\right]$ **(AU-2014)**

$$L\left[\frac{\sin t}{t}\right] = \int_s^\infty L(\sin t)ds = \int_s^\infty \frac{1}{s^2+1}ds = \left[\tan^{-1}(s)\right]_s^\infty = \left[\tan^{-1}(\infty) - \tan^{-1}(s)\right]$$

$$= \frac{\pi}{2} - \tan^{-1}(s) = \cot^{-1}(s)$$

11. Find the Laplace transform of the function $f(t) = \frac{1-e^{-t}}{t}$ **(AU-2013)**

$$L(f(t)) = L\left[\frac{1-e^{-t}}{t}\right] = \int_s^\infty L(1-e^{-t})ds = \int_s^\infty L(1) - L(e^{-t})ds = \int_s^\infty\left[\frac{1}{s} - \frac{1}{s+1}\right]ds$$

$$= \left[\log s - \log(s+1)\right]_s^\infty$$

$$= \log\left[\frac{s+1}{s}\right]$$

12. Verify initial value theorem for $f(t)=1+e^{-t}(\sin t+\cos t)$ **(AU-2012)**

Initial value theorem is ,if $L\left[f(t)\right]$=F(s),then $\underset{t\to0}{Lt}\,f(t) = \underset{t\to\infty}{Lt}\,sF(s)$

F(s)=L(1+)= $L\left[1+e^{-t}(\sin t+\cos t)\right]$

F(s)= $\frac{1}{s} + L[\sin t + \cos t]_{s\to s+1}$

F(s)= $\frac{1}{s} + \frac{s+2}{(s+1)^2+1}$

$\underset{t\to0}{Lt}\,f(t) = \underset{t\to0}{Lt}\; 1+e^{-t}(\sin t + \cos t) \quad =2$

$$\lim_{t\to\infty} sF(s) = \lim_{s\to\infty} s\left[\frac{1}{s} + \frac{s+2}{(s+1)^2+1}\right] = \lim_{s\to\infty}\left[1 + \frac{s^2\left(1+\frac{2}{s}\right)}{s^2\left(1+\frac{2}{s}+\frac{2}{s^2}\right)}\right] = 2$$

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s) = 2$$

Hence the initial value theorem is verified.

13. Verify Initial value theorem for the function f(t) = ae^{-bt} **(AU-2013)**

$$f(t) = ae^{-bt}, \; F(s) = L[f(t)] = L[ae^{-bt}] = \frac{a}{s+b}$$

Initial value theorem: $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$

L.H.S = $\lim_{t\to 0} f(t) = \lim_{t\to 0} ae^{-bt} = a$

R.H.S $\lim_{s\to\infty} sF(s) = \lim_{s\to\infty} s\left(\frac{a}{s+b}\right) = \lim_{s\to\infty} \frac{as}{s+b}$

$$= \lim_{s\to\infty} \frac{as}{s\left(1+\frac{b}{s}\right)} = \lim_{s\to\infty} \frac{a}{1+\left(\frac{b}{s}\right)} = a$$

Hence the initial value theorem is verified. **(AU-2013)**

14. If $L\left(e^{-t}\cos^2 t\right) = F(s)$, find $\lim_{s\to 0} sF(s)$

$$F(s) = L(e^{-t}\cos^2 t) = L\left[\cos^2 t\right]_{s\to s+1}$$

$$= L\left[\frac{1+\cos 2t}{2}\right]_{s\to s+1}$$

$$= \frac{1}{2}L[1+\cos 2t]_{s\to s+1} = \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+4}\right]_{s\to s+1}$$

$$L(e^{-t}\cos^2 t) = \frac{1}{2}\left[\frac{1}{s+1} + \frac{s+1}{(s+1)^2+4}\right]$$

(AU-2010)

15. Define periodic function with an example.

A function f(t) is said to have a period T or to be periodic with period T if for all t, f(t+T) = f(t) where T is a positive constant. The least value of T>0 is called the period of f(t) .

Eg. Consider

$$f(t) = \sin t$$

$$f(t+2\pi) = \sin(t+2\pi) = \sin t$$

$$i.e.\; f(t) = f(t+2\pi) = \sin t$$

$$\therefore \sin t \text{ is a periodic function with period } 2\pi$$

(AU-2014)

16. Evaluate $L^{-1}\left[\frac{1}{s^2+6s+13}\right]$

$$L^{-1}\left[\frac{1}{s^2+6s+13}\right] = L^{-1}\left[\frac{1}{s^2+6s+9+4}\right] = L^{-1}\left[\frac{1}{(s+3)^2+2^2}\right] = e^{-3t}\frac{\sin 2t}{2}$$

(AU-2012)

17. Find the Laplace inverse transform of $\frac{1}{(s+1)(s+2)}$

$$\frac{1}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$$

A=1 and B=1(using partial fraction)

$$L^{-1}\left[\frac{1}{(s+1)(s+2)}\right] = L^{-1}\left[\frac{1}{s+1}\right] + L^{-1}\left[\frac{1}{s+2}\right]$$

$$L^{-1}\left[\frac{1}{(s+1)(s+2)}\right] = e^{-t} + e^{-2t}$$

18. Find the inverse Laplace transform of $\log\left(\frac{s+1}{s-1}\right)$ **(AU-2012)**

We know that $L^{-1}[F(s)] = \frac{-1}{t} L^{-1}[F'(s)]$

$$L^{-1}\left[\log\left(\frac{s+1}{s-1}\right)\right] = \frac{-1}{t} L^{-1}\left[\frac{d}{ds}\left[\log\left(\frac{s+1}{s-1}\right)\right]\right]$$

$$= \frac{-1}{t} L^{-1}\left[\frac{d}{ds}[\log(s+1) - \log(s-1)]\right]$$

$$= \frac{-1}{t} L^{-1}\left[\frac{1}{s+1} - \frac{1}{s-1}\right] = \frac{-1}{t}[e^{-t} - e^{t}] = \frac{2}{t}\sinh t$$

19. **Find the Laplace transform of** $\int_0^t te^{-t}\,dt$ **(AU-2015)**

$$L\left[\int_0^t te^{-t}\,dt\right] = \frac{1}{s} L[te^{-t}]$$

$$= \frac{1}{s}\left[-\frac{d}{ds} L[e^{-t}]\right]$$

$$= \frac{1}{s}\left[-\frac{d}{ds}\left(\frac{1}{s+1}\right)\right]$$

$$= \frac{-1}{s}\left(\frac{1}{(s+1)^2}\right) = \frac{-1}{s(s+1)^2}$$

(AU-2014)

20. Find the inverse Laplace transform of $\frac{e^{-\pi s}}{(s-1)^2}$

$$L^{-1}\left(\frac{1}{s^2}\right) = t \quad and \quad L^{-1}\left(\frac{1}{(s-1)^2}\right) = te^{t}$$

$$L\left(\frac{e^{-\pi s}}{(s-1)^2}\right) = (t-\pi)e^{(t-\pi)}$$

PART-B

1. a .Find $L\left(t^2 e^{-3t} \sin 2t\right)$ **(AU-2013)(8)**

b. Find $L\left[\frac{\cos at - \cos bt}{t}\right]$ **(AU-2015)(8)**

2. a. Find the Laplace transform of the square-wave function (or Meoander function) of Period a defined as $f(t) = \begin{cases} 1, & when 0 < t < \frac{a}{2} \\ -1, & when \frac{a}{2} < t < a \end{cases}$ **(AU-2013)(8)**

b. Find the Laplace transform of the following triangular wave function given by

$f(t)=\begin{cases} t & 0\le t\le c \\ 2c-t & c\le t\le 2c \end{cases}$ and $f(t+2c)=f(t)$. (AU-2015)(8)

3. a. Find the Laplace transform of the periodic function defined on the interval $0\le t\le 1$ by

$f(t)=\begin{cases} 1, & 0\le t\le \frac{1}{2} \\ -1 & \frac{1}{2}\le t<1 \end{cases}$ and $f(t+1)=f(t)$. (AU-2014)(8)

b. Find the Laplace transform of $f(t)=t^2$, $0<t<2$ and $f(t+2)=f(t)$ for all $t>0$ (AU-2013)(8)

4. a. Find the inverse Laplace transform of $\frac{4s+7}{s^2(2s+3)(3s+5)}$ (AU-2013)(8)

b. Find $L^{-1}(s/(s^2+1)(s^2+4))$ (AU-2015)(8)

5.a. Find the Laplace transforms of the following functions 1)e^ttcost 2)1-cost/t п/w (AU-2014)(8)

b. Find the Laplace transform of f(t)= $te^{-3t}\cos 2t$. (AU-2014)(8)

6. a. Find the Laplace transforms of $f(t)=\begin{cases} t & 0<t<a \\ 2a-t & a<t<2a \end{cases}$, where $f(t+2a)=f(t)$ (AU-2014)(8)

b. Find the Laplace transform of f(t) where

$f(t)=\begin{cases} \sin\omega t & \text{for } 0<t<\frac{\pi}{\omega} \\ 0 & \text{for } \frac{\pi}{\omega}<t<\frac{2\pi}{\omega} \end{cases}$, $f\left(t+\frac{2\pi}{\omega}\right)=f(t)$ (AU-2014)(8)

7. a. Find the Laplace transform $f(t)=\begin{cases} \cos t & 0<t<\pi \\ 0 & \pi<t<2\pi \end{cases}$, $f(t+2\pi)=f(t)$

b. Find $L^{-1}\left[\frac{3s^2+16s+26}{s(s^2+4s+13)^2}\right]$ (AU-2013)(8)

8.a .Find $L^{-1}\left[\frac{s^2}{(s^2+a^2)^2}\right]$ and find $L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right]$ hence find $L^{-1}\left[\frac{1}{(s^2+9s+13)^2}\right]$ (AU-2013)(8)

b. Use convolution theorem to find the inverse of $\frac{s}{(s^2+4)(s^2+9)}$ (AU-2013)(8)

9. a. Find the Laplace transform of $f(t)=\frac{\cosh t\cos t}{t}$ and $g(t)=\frac{\sin\sqrt{t}}{t}$ (AU-2013)(8)

b. Using convolution theorem to find the inverse Laplace transform of the function

$\frac{s}{(s^2+1)^2}$ (AU-2014)(8)

10.a. Using convolution theorem to find the inverse Laplace transform of the function

$\frac{s^2}{(s^2+a^2)(s^2+b^2)}$ (AU-2014)-2(8)

b .Using convolution, solve the initial value problem,y''+9y=sin3t, y(0)=0,y'(0)=0.

11. a. Verify initial and final value theorem for $f(t) = 1 + e^{-t}(\sin t + \cos t)$ **(AU-2014)(8)**
    b. Verify initial and final value theorem for $f(t) = 1 + e^{-2t}$

12.a. Solve $y'' + y = t^2 + 2t$, $y(0) = 4$, $y'(0) = -2$ by using Laplace transform. **(AU-2013)(8)**
    b. Solve the differential equation $y'' - 3y' + 2y = 4t + e^{3t}$ where y(0)=1 and y^1 **(0)=-1** using Laplace transforms **(AU-2015)(8)**

13.a Solve $y'' - 6y' + 9y = t^2 e^{3t}$, $y(0) = 2$, $y'(0) = 6$ by Laplace transform method **(AU-2014)(8)**
    b. Solve the following differential equation ,using Laplace transform
    $y'' + 2y' + 2y = 8e^{t}\sin t$, $y(0) = y'(0) = 0$ **(AU-2013)(8)**

14.a. Using Laplace Transform , solve $\frac{d^2y}{dt^2} + 4y = \sin 2t$ given y(0) =3, y'(o)=4 **(AU-2014)(8)**
    b.Use Laplace transform to solve $(D^2-3D+2)y = e^{3t}$ with y(0)=1 ,$y^1(0)=0$ **(AU-2014)(8)**

15.a. Using Laplace transform method, solve $d^2y/dt^2+9y=18t$ given that $y(0)=0, y(\pi/2)=0$.
    b. **U**sing Laplace transform ,find the solution of $y' + 3y + 2\int_0^t y\,dt = t$ $y(0) = 0$

UNIT-III
ANALYTIC FUNCTIONS
PART- A

1. State the basic difference between the limit of a function of a real variable and that of a complex variable. (AU2012)

Real variable	Complex Variable
Limit takes along x axis and y axis or parallel to both axis	Limit takes along any path (straight or curved)

2. State the necessary condition of Cauchy-Riemann equations **(AU-2011)**

The necessary condition for $f(z) = u(x, y) + iv(x, y)$ to be analytic in a region R are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

3. Write 2-D Laplace equations in polar coordinates. **(AU-2011)**

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = 0$$

4. Show that the function $f(z) = \bar{z}$ is nowhere differentiable. **(AU-2014)-2**

Given $f(z) = \bar{z} = x - iy$

$u = x, v = -y$

$\frac{\partial u}{\partial x} = 1, \frac{\partial v}{\partial x} = 0$

$\frac{\partial u}{\partial y} = 0, \frac{\partial v}{\partial y} = -1$

$u_x \neq v_y$, C-R equations are not satisfied anywhere. Hence

$f(z) = \bar{z}$ is nowhere differentiable.

5. Find the constants a,b if $f(z) = x + 2ay + i(3x + by)$ is analytic **(AU-2013)**

$f(x) = x + 2ay + i(3x + by)$

$u = x + 2ay \quad and \quad v = (3x + by)$

Where $\frac{\partial u}{\partial x} = 1, \frac{\partial u}{\partial y} = 2a$

$\frac{\partial v}{\partial x} = 3, \quad \frac{\partial v}{\partial y} = b$

We know that by CR equations

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad and$

$a = \frac{3}{2}, \quad b = 1$

6. If u+iv is analytic , show that v –iu &-v +iu are also analytic **(AU-2013)**

Given u+iv is analytic

C –R equations are satisfied $u_x=v_y$ …….. (1)

$u_y = -v_x$ ....(2)

Since the derivatives of u & v exis it is therefore continuous

Now to prove v –iu & -v+iu are also analytic, we should prove that

(i) $v_x = -u_y$ & $v_y = u_x$ &

(ii) $v_x = u_v$ & $v_y = u_x$

(iii) u_x, u_y, v_x, v_y are all continuous . Results (i) & (ii follows from (1) & (2) . since the derivatives of u &v exists from (1) and (2) , the derivatives of u and v should be continuous .

Hence the result

7. Find the value of a,b,c,d so that the function $f(z) = (x^2 + axy + by^2) + i(cx^2 + dxy + y^2)$ **may be analytic** **(AU-2013)**

$f(z) = (x^2 + axy + by^2) + i(cx^2 + dxy + y^2)$

$u = x^2 + axy + by^2 \quad , v = cx^2 + dxy + y^2$

$u_x = 2x + ay \quad , v_x = 2cx + dy$

$u_y = ax + 2by \ , v_y = dx + 2y$

$f(z)$ is analytic , $u_x = v_y$ and $u_y = -v_x$

$a = 2, b = -1, c = -1, d = 2$

8. State whether or not $f(z) = \bar{z}$ **is an analytic function** **(AU-2012)-2**

$w = f(z) = \bar{z}$

$u + iv = x - iy \Rightarrow u = x$ and $v = y$

$u_x = 1, \quad v_x = 0$

$u_y = 0, \quad v_y = -1$

$u_x \neq v_y$

Hence CR equations are not satisfied

∴ The function $f(z)$ is nowhere analytic

9. Verify whether or not $f(z) = e^x(\cos y - i\sin y$ **is analytic** **(AU-2014)**

$u = e^x \cos y$ and $v = -e^x(\sin y)$

$\frac{\partial u}{\partial x} = e^x \cos y$ and $\frac{\partial v}{\partial x} = -e^x \sin y$

$\frac{\partial u}{\partial y} = -e^x \sin y$ and $\frac{\partial v}{\partial y} = -e^x \cos y$

$\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x}$

CR equations are not satisfied. It is not an analytic function.

10. S.T $f(z) = e^x \sin y$ **is harmonic** **(AU-2014)**

$u_x = e^x \sin y, \quad u_y = e^x \cos y$

$u_{xx} = e^x \sin y \quad u_{yy} = -e^x \sin y$

$u_{xx} + u_{yy} = e^x \sin y - e^x \sin y = o$

$f(z) = e^x \sin y$ is **harmonic**

11. If $f(z)$ **is an analytic function whose real part is constant, Prove that** $f(z)$ **is a constant function.** **(AU-2012)**

Let $f(z) = u(x, y) + iv(x, y)$ be an analytic function

Therefore by CR equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$

Given u =constant

To prove $f(z)$ is a constant

$u = c$

$\frac{\partial u}{\partial x} = 0, \quad \frac{\partial u}{\partial y} = 0$

By CR equation $\frac{\partial u}{\partial x} = 0 \Rightarrow \frac{\partial v}{\partial y} = 0$ and $\frac{\partial u}{\partial y} = 0 \Rightarrow \frac{\partial v}{\partial x} = 0$

$\therefore f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 0 + i0$

$f'(z) = 0 \Rightarrow f(z) = c$

$f(z)$ is a constant.

12. Find the image of the line x=k under the transformation $w = \frac{1}{z}$ **(AU-2013)**

Given $w = \frac{1}{z}$

i.e., $z = \frac{1}{w} = \frac{1}{u+iv} = \frac{u-iv}{(u+iv)(u-iv)} = \frac{u-iv}{u^2+v^2}$

$x + iy = \frac{u}{u^2+v^2} + i\left(\frac{-v}{u^2+v^2}\right)$

i.e., $x = \frac{u}{u^2+v^2}$ ....(1), $y = \frac{-v}{u^2+v^2}$ ....(2)

Given x=k in the z plane

$k = \frac{u}{u^2+v^2}$ .by(1),

$k(u^2+v^2) = u$

$u^2+v^2 - \frac{1}{k}u = 0$

$\left(u - \frac{1}{2k}\right)^2 + v^2 - \frac{1}{4k^2} = 0$

$\left(u - \frac{1}{2k}\right)^2 + v^2 = \frac{1}{4k^2}$ which is a circle whose centre is $\left(\frac{1}{2k}, 0\right)$ and radius $\frac{1}{2k}$

13. Find the map of the circle $|z| = 3$ under the transformation $w = 2z$

$w = 2z$

$u + iv = 2(x + iy)$

$u = 2x,\ v = 2y \Rightarrow x = \frac{u}{2},\ y = \frac{v}{2}$

Given $|z| = 3 \Rightarrow |x + iy| = 3 \Rightarrow x^2 + y^2 = 9$

$\therefore \left(\frac{u}{2}\right)^2 + \left(\frac{v}{2}\right)^2 = 9 \Rightarrow u^2 + v^2 = 36$

Hence the image of $|z| = 3$ in the z-plane is transformed into $u^2 + v^2 = 36$

(AU-2012)

14.Find the image of the circle z **= 2 under the transformation** $\omega = 3z$ **(AU-2012)**

$$w = 3z$$

$$u + iv = 3(x + iy)$$

or $cz^2 + ($

$u = 3x,\ v = 3y \Rightarrow x = \frac{u}{3},\ y = \frac{v}{3}$

Given $|z| = 2 \Rightarrow |x + iy| = 2 \Rightarrow x^2 + y^2 = 4$

$\therefore \left(\frac{u}{3}\right)^2 + \left(\frac{v}{3}\right)^2 = 4 \Rightarrow u^2 + v^2 = 36$

Hence the image of $|z| = 2$ in the z-plane is transformed into $u^2 + v^2 = 36$ in the w-plane under the transformation $w = 3z$

15. Find the image of the circle $|z| = \lambda$ under the transformation $w = 5z$

$w = 5z$

$u + iv = 5(x + iy)$

$u = 5x,\ v = 5y \Rightarrow x = \frac{u}{5},\ y = \frac{v}{5}$

Given $|z| = \lambda \Rightarrow |x + iy| = \lambda \Rightarrow x^2 + y^2 = \lambda^2$

$\therefore \left(\frac{u}{5}\right)^2 + \left(\frac{v}{5}\right)^2 = \lambda^2 \Rightarrow u^2 + v^2 = (5\lambda)^2$

Hence the image of $|z| = \lambda$ in the z-plane is transformed into $u^2 + v^2 = (5\lambda)^2$ in the w-plane under the transformation $w = 5z$

16.Define critical point of a transformation

A point z_0 at which the mapping w=f(z) is not conformal is called the critical point .

17. **Find the invariant points of the transformation $f(z) = z^2$**

$f(z) = z^2$

$w = z^2$,

$z = z^2$

$z^2 - z = 0$

$z(z - 1) = 0$

$z = 0,\ z = 1$

The invariants points are z=0,z=1.

18. Find the critical points of the transformation $w = 1 + \frac{2}{z}$

$z = 1 + \frac{2}{z} \quad z^2 - z - 2 = 0 \quad (z - 2)(z + 1) = 0$

$z = 2,\ z = -1$

Critical points are z=2,-1

19. **Find the invariant points of the transformation $\omega = \frac{2z + 6}{z + 7}$**

The invariant points are given by $z = \frac{2z + 6}{z + 7}$

$z^2 + 7z - 2z - 6 = 0 \Rightarrow z^2 + 5z - 6 = 0$

$(z + 6)(z - 1) = 0$

$z = -6, 1$

20. Prove that a bilinear transformation has atmost two fixed points.

The fixed points of the transformation $w = \frac{az + b}{cz + d}$ is obtained from $z = \frac{az + b}{cz + d}$

(AU-2011) (AU-2013)

(AU-2013)

(AU-2010)

(AU-2014) (AU-2012)

These points are two in number unless the discriminant is zero in which case the number of points is one.

21.Show that $|z|^2$ is not analytic at any **point.** **(AU-2015)**

$f(z) = |z|^2 = u+iv = x^2+y^2$ where $u = x^2+y^2$ and $v=0, u_x=2x, u_y=2y, v_x=0, v_y=0$

hence f(z) is not analytic.

PART-B

1. a. If $f(z)$ is an analytic function of z, prove that $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$ **(AU-2013)(8)**

b. show that a harmonic function u satisfies the formal differential equation $\frac{\partial^2 u}{\partial z \partial \bar{z}} = 0$ and hence P.T $\log|f^1(z)|$ is harmonic, where f(z) is a regular function. **(AU-2015)(8)**

2. a. Show that the function $u = e^{-x}(x\cos y + y\sin y)$ is harmonic function. Hence find the corresponding analytic function $f(z) = u + iv$ **(AU-2014)(8)**

b. Determine the analytic function $w = u + iv$ given that $u = e^{-2x}(x\cos y + y\sin y)$ **(AU-2015)(8)**

3. a Prove that $u = e^{-y}\cos x$ and $v = e^{-x}\sin y$ satisfy Laplace equations but that u+iv is not an analytic function of z.

b. Find if $\Phi = (x-y)(x^2+4xy+y^2)$ can represent the equipotential surface for an electric field. Find the corresponding complex potential $\omega = \phi + i\psi$ and also ψ **(AU-2013)(8**

4.a. Find the analytic function $f(z) = u + iv$ where $v = 3r^2\sin 2\theta - 2r\sin\theta$. Verify that u is a harmonic function. **(AU-2013)(8)**

b. Determine the analytic function whose real part is $\frac{\sin 2x}{\cosh 2y - \cos 2x}$ **(AU-2014) (8)**

5.a. Prove that the function $u = e^x(x\cos y - y\sin y)$ satisfies Laplace's equation and find the corresponding analytic function $f(z) = u + iv$ **(AU-2013)(8)**

b. Prove that the real and imaginary parts of an analytic function are harmonic function**.** **(AU-2014) (8)**

6.a. Find the analytic function $w = u + iv$ if $e^{2x}(x\cos 2y - y\sin 2y)$ and hence find u $f(z) = u(x, y) + iv(x, y)$ given that **(AU-2013)(8)**

b. Find the analytic function $u - v = e^x(\cos y - \sin y)$ **(AU-2014)(8)**

7. a . If $u = x^2 - y^2$ and $v = \frac{-y}{x^2+y^2}$ prove that both u and v satisfy Laplace equations, but $u + iv$ is not a regular function of z **(AU-2013)(8)**

b. Find the image of the circle $|z| = 2$ under the transformation $\omega = z + 3 + 2i$ **(AU-2013)(8)**

8.a. Find the image of w plane of the region of the z-plane bounded by the straight line x=1,y=1 and x+y=1 under the transformation $w = z^2$ **(AU-2013)(8)**

b.Find the image in the w-plane of the infinite strip $1/4 \le y \le 1/2$ under the transformation w=1/z **(AU-2015)(8)**

9.a. Prove that $w = \frac{z}{1-z}$ maps the upper half of the z-plane to the upper half of the w-plane and also find the image of the unit circle of the z plane. **(AU-2013)(8)**

b .Find the image of the circle $|z - 3i| = 3$ and the region $1 < x < 2$ under the map $w = \frac{1}{z}$

10 .a. Find the image of $|z+2i|=2$ under the transformation w=1/z.

b. Find the image of the following regions under the transformation w=1/z.

i) the half plane x>c when c>0

ii) the half plane y>c when c<0

11.a. S.T under the mapping w= i-z/i+z ,the image of the circle $x^2+y^2<1$ is the entire half of the w-plane to the right of the imaginary axis

b. Find the image of the region bounded by the lines x=0,y=0,and x+y =1 under the mappings $w = e^{\frac{i\pi}{4}}$ and $w = z + (2 + 3i)$ **(AU-2014)(8)**

12.a Show that the image of the hyperbola $x^2 - y^2 = 1$ under the transformation w = $\frac{1}{z}$ is the lemniscates $r^2 = \cos 2\theta$ ***(AU-2012)*(8)**

b. Find the Bilinear transformation which maps z= 0,z=1,z=∞ into the points w=i, w=1,w=-i **(AU-2013)(8)**

13.a. Find the bilinear transformation that maps 1,i,and -1 of the z-plane onto 0,1 and ∞ on the w- plane.Also find the image of the unit circle of the z plane. **(AU-2014) (8)**

b. Find the Bilinear transformation that maps the points z=∞,1,0 onto the points w=o,i, ∞ respectively **(AU-2012)(8)**

14.a. Find the Bilinear transformation that maps the points z=1,i,-1 into the points w=,0,1, ∞ respectively. Find also the pre-image of $|w|$ =1 under this bilinear transformation. **(AU-2014)(8)**

b. Find the bilinear transformation that maps the points z=0,-1,i into the points w= i,0,∞ respectively. **(AU-2015)(8)**

15. a. Find the bilinear transformation that maps the points 1+i,-i,2-i of the z- plane into the points 0,1,i of the w-plane.

b. Find the bilinear transformation that maps the points z=i,-1,1 into the points w=0,1,∞ respectively.

UNIT-IV
COMPLEX INTEGRATION
PART-A

1. State Cauchy's integral theorem **(AU-2015)**

If f (z) is analytic inside and on a closed curve c of a simply connected region R and if 'a' is any point within c, then $f(a) = \frac{1}{2\pi i}\int_c \frac{f(z)}{z-a}\,dz$, the integration around C being taken in the positive direction .

2. Evaluate $\int_C \frac{e^{-z}}{z^2}dz$ **,where C is a circle** $|z| = 1$ **.** **(AU-2012)**

We know that $\int_c \frac{f(z)}{z-a}dz = 2\pi i f(a)$, $\int_c \frac{e^{-z}}{z^2}dz = \int_c \frac{e^{-z}}{(z-0)^2}$

Here $f(z) = e^{-z}$, a= 0 is lies inside $|z| = 1$

By cauchy's integral formula we get

$\int_C \frac{e^{-z}}{z^2}dz = 2\pi i f'(a) = 2\pi i(-1) = -2\pi i$ z

3. **Evaluate** $\int_C \frac{z^2+1}{z^2-1}dz$ **where C is a circle of unit radius and centre at z=i.** **(AU-2013)**

$|z - i| = 1$

The poles z=1,z=-1 lies outside the circle

$\therefore \frac{z^2+1}{z^2-1}$ is analytic inside $|z - i| = 1$

By Cauchy's theorem, $\int_C \frac{z^2+1}{z^2-1}dz$ **=0**

4. Evaluate $\int_z \sec z\,dz$ **where c is the unit circle** $|z| = 1$ **(AU-2014)**

$\int_z \sec z\,dz = \int_z \frac{1}{\cos z}dz$

The pole are given by the solution of cos z = 0

$i.e., z = (2n+1)\frac{\pi}{2}, n = 0,1,2,.......$

$z = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, ...$

Hence all the poles lies outside $|z| = 1$, secz is analytic with $|z| = 1$

By Cauchy's theorem $\int_z \sec z\,dz = 0$

5. Evaluate $\int_C \frac{3z^2+7z+1}{(z+1)}dz$ **where C is |z|=1/2** **(AU-2013)**

$\int_C \frac{3z^2+7z+1}{z-(-1)}dz$ Here z=-1 lies outside c.

- $f(z)$ is analytic inside and on c
- $f'(z)$ is continuous inside c.

Hence by cauchy' s theorem $\int_c f(z)dz = 0$

6. State Taylor's theorem. **(AU-2011)**

A function f(z) , is analytic inside a circle C with centre at a ,can be expanded in the series

$$f(z)=f(a)+f'(a)(z-a)+\frac{f''(a)}{2!}(z-a)^2+\frac{f'''(a)}{3!}(z-a)^3+\dots+\frac{f^{(n)}(a)}{n!}(z-a)^n+\dots \text{ to } \infty$$

Which is convergent at every point inside C

7. Find the Taylor series of the function f(z)=sinz about z=π/4 **(AU-2013)**

$f(z)=\sin z$

$f'(z)=\cos z$

$f''(z)=-\sin z$

$f'''(z)=-\cos z$

Here $a=\frac{\pi}{4}$, $f(\frac{\pi}{4})=\sin(\frac{\pi}{4})=\frac{1}{\sqrt{2}}$

$f'(\frac{\pi}{4})=\cos(\frac{\pi}{4})=\frac{1}{\sqrt{2}}$, $f''(\frac{\pi}{4})=-\sin(\frac{\pi}{4})=-\frac{1}{\sqrt{2}}$

$f'''(\frac{\pi}{4})=-\cos(\frac{\pi}{4})=-\frac{1}{\sqrt{2}}$

Taylor's series is $f(z)=f(\frac{\pi}{4})+\frac{z-\frac{\pi}{4}}{1!}f'(\frac{\pi}{4})+\frac{(z-\frac{\pi}{4})^2}{2!}f''(\frac{\pi}{4})+\dots\dots$

8. Find the Laurent's series for the function $f(z)=z^2e^{1/z}$ about z=0 **(AU-2013)**

$$z^2e^{\frac{1}{z}}=z^2\left[1+\frac{1}{z}+\frac{1}{2!}\frac{1}{z^2}+\dots\dots\right]$$

$$=z^2+z+\frac{1}{2}+\dots\dots$$

9. Define singular point. **(AU-2012)**

A point $z=z_0$ at which a function f(z)fails to be analytic is called a singular point or singularity of f(z).

10. Identify the types of singularities of the following function $f(z)=e^{\frac{1}{(z-1)}}$ **(AU-2009)**

Here z=1 is a singular point

At z=1 , we get $f(z)=e^{\frac{1}{0}}=\infty$ which is not defined.

Also z=1 is not a pole or removable singularity

Z=1 is an essential singularity.

11. **Discuss the nature of the singularities of the function** $f(z)=\frac{\sin z}{z}$ **(AU-2012)-2**

Poles of f(z) are obtained by equating the denominator to zero

i.e $f(z)=\frac{\sin z}{z}$

z=o is a pole of order 1

sinz=0

z=n π where n= 0, $n=0,\pm1,\pm2\dots$

12.Identify the type of singularity of function sin(1/1-z) **(AU-2015)**

$\text{Sin}(1/1-z)=1/1-z-1/3!(1/1-z)^3+1/5!(1/1-z)^5$

The RHS is the Laurent series with infinite number of terms about the singular part z=1,z=1 is an essential singularity of f(z).

$\cos z$

$\sin z$

13. Find the nature of the singularity z=0 of the function $f(z) = \dfrac{1-\cos z}{z^2}$	Poles

of f(z) are obtained by equating the denominator to zero

i.e $f(z) = \dfrac{1-\cos z}{z^2}$ **(AU-2011)**

$z^2=0$ is a pole of order 2

14. State Cauchy's residue theorem.

If f(z) be an analytic at all points inside and on a simple closed curve C, except for a finite number of isolated singularities $z_1, z_2, z_3, \ldots, z_n$ inside c then (AU-2014)

$\int f(z)dz = 2\pi i$ [sum of the residues of f(z) at $z_1, z_2, z_3, \ldots, z_n$] $= 2\pi i \sum_{i=1}^{n} R_i$

where R_i is the residue of f(z) at z=zi

15. If f(z)= $\frac{-1}{z-1} - 2\left[1+(z-1)+(z-1)^2+\cdots\right]$**,find the residue of f(z) at z=1** **(AU-2012)**

Residue of $f(z)$ at $z=1$ is -1 (the coefficient of $\frac{1}{z-1}$)

16. Find the residue of $\frac{1-e^{2z}}{z^4}$ **at z=0** **(AU-2013)**

Given $f(z)=\frac{1-e^{2z}}{z^4}$

Here z = 0 is a pole of order 4

$$\text{Re}\, s(z=0) = \frac{1}{3!}\underset{z\to 0}{Lt}\frac{d^3}{dz^3}\left[(z-0)^4\frac{1-e^{2z}}{z^4}\right]$$

$$= \frac{1}{6}\underset{z\to 0}{Lt}\frac{d^3}{dz^3}\left[1-e^{2z}\right] = -\frac{4}{3}$$

17. Find the residue of the function f(z)= $\frac{4}{z^3(z-2)}$ **at a simple pole** **(AU-2012)**

$$f(z)=\frac{4}{z^3(z-2)}=\frac{4}{(z-0)^3(z-2)}$$

Here z = 0 is a pole of order 3 and z=2 is a pole of order 1

$$\text{Re}\, s(z=0)=\frac{1}{2!}\underset{z\to 0}{Lt}\frac{d^2}{dz^2}\left[(z-0)^3\frac{4}{(z-0)^3(z-2)}\right]$$

$$=\frac{1}{2!}\underset{z\to 0}{Lt}\frac{d^2}{dz^2}\left[\frac{4}{(z-2)}\right]=\frac{1}{2}\underset{z\to 0}{Lt}\left[\frac{8}{(z-2)^3}\right]=\frac{1}{2}$$

$$\text{Re}\, s(z=2)=\underset{z\to 2}{Lt}\left[(z-2)\frac{4}{(z-0)^3(z-2)}\right]=\frac{1}{8}$$

(AU-2012)

18. Find the residue of $f(z)=\frac{z+1}{(z-1)(z-2)}$ **at z=2**

$$\text{Re}\, s(z=2)=\underset{z\to 2}{Lt}\left[(z-2)\frac{z+1}{(z-1)(z-2)}\right]=3$$

(AU-2010)

19. Find the resuidue of cot z at the pole z=0.

f(z)=cotz= ____ Poles of f(z) are sinz = 0 = sin n π

z = n π, where n = 0,±1, ±2,….

$$[\operatorname{Res} f(z)]_{z=n\pi} = \lim_{z\to n\pi} \frac{(z-n\pi)\cos z}{\sin z} = \lim_{z\to n\pi} \frac{-(z-n\pi)\sin z + \cos z(1)}{\cos z} \text{ (by L' Hospital rule)}$$

$$[\operatorname{Res} f(z)]_{z=n\pi} = 1$$

20. Determine the residue of $f(z) = \frac{z^2}{(z-1)^2(z+2)}$ **at z=1** **(AU-2012)**

Given $f(z) = \frac{z^2}{(z-1)^2(z+2)}$

Here z =1 is a pole of order 2

$$\operatorname{Re} s[z = z_0] = Lt_{z\to z_0} \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}[z - z_0]^m f(z)$$

$$\operatorname{Re} s[z=1] = lt_{z\to 1} \frac{d}{dz}\left[(z-1)^2 \frac{z^2}{(z-1)^2(z+2)}\right] = Lt_{z\to 1}\frac{d}{dz}\left(\frac{z^2}{z+2}\right)$$

$$= f(z) = \frac{5}{9}$$

21. Find the residue of $f(z) = \frac{50z}{(z+4)(z-1)^2}$ **at z = 1** **(AU-2009)**

Z = 1 is a pole of order 2

$$\operatorname{Re} s[f(z)]_{z=1} = Lt_{z\to 1}\frac{d}{dz}\left[(z-1)^2 \frac{50z}{(z+4)(z-1)^2}\right]$$

$$= Lt_{z\to 1}\left[\frac{(z+4)50 - 50z}{(z+4)^2}\right] = \frac{250-50}{25} = 8$$

22. Evaluate $\int_c \frac{e^z}{z-1}dz$ **if c is |z|=2** **(AU-2010)**

Z=1 is a pole of order 1 which lies inside |z|=2

$$\int_c \frac{e^z}{z-1} dz = 2\pi i f(1)$$

$$= 2\pi i e$$

PART-B

1. a .Evaluate $\int_C \frac{z}{(z-1)(z-2)}dz$ here C is $\left|z-2\right| = \frac{1}{2}$ by using Cauchy's integral formula.

(AU-2012)(8)

b .Evaluate $\int \frac{7z-1}{z^2-3z-4} dz$ over the curve C:$x^2+4y^2=4$ using Cauchy's integral formula.

(AU-2013)(8)

2.a. Evaluate $\int_C \frac{z+1}{(z^2+2z+4)^2} dz$ where c is the circle $\left|z+1+i\right| = 2$ by Cauchy's integral formula.

(AU-2013)(8)

b. Evaluate $\int_C \frac{z+4}{z^2+2z+5} dz$ where C is the circle |z+1+i|=2 using Cauchy's integral formula.

(AU-2013)(8)

3.a.Using Cauchy's integral formula ,evaluate $\int_C \frac{e^z}{(z+1)(z+2)} dz$ where C is |z|=3. **(8)**

b. If $f(a) = \int \frac{13z^2+27z+15}{z-a} dz$ where c is the circle $\left|z\right| = 2$ then find

$f(3), f'(1-i), f''(1-i)$ *and* $f(1-i)$ **(AU-2014)(8)**

4.a. Evaluate $\int_C \frac{z^3}{(2z+i)^3} dz$ where c is the unit circle $|z| = 1$ **(8)**

b. Obtain Taylor's series for $f(z) = \frac{2z^3}{z(z+1)^3}$ about z=i **(AU-2013)(8)**

5.a. Evaluate $f(z) = \frac{1}{(z+1)(z+3)}$ in Laurent series valid for the regions |z|>3 and 1<|z|<3 **(AU-2013)(8)**

b. Find the Laurent's series expansion of $f(z) = \frac{7z-2}{(z-2)(z+1)}$ valid in the region

$|z+1| < 1$ and $|z+1| > 3$ $z^2 - 1$ **(8)**

6.a. Expand the function $f(z) = \frac{}{z^2 + 5z + 6}$ in Laurent's series |z|>3 **(AU-2013)(8)**

b. Obtain the Laurent's series expansion of $f(z) = \frac{z^2 - 1}{(z+2)(z+3)}$ in $2 < |z| < 3$ **(AU-2015)(8)**

7.a.Expand $f(z) = \frac{1}{}$ $z^2 - 4z + 3$ as the Laurent's series expansion of **1<** $|z| < 3$ **(AU-2014)(8)**

b. Obtain the Laurent's series expansion of $f(z) = \frac{1}{z - z^2}$ in the region 1<|z+1|<2 and|z+1|>2.

(AU-2014)(8)

8.a. Evaluate $\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz$ where C is |z|=3 Using Cauchy's Residue theorem **(AU-2013) (8)**

b. Using Cauchy's residue theorem evaluate $\int_C \frac{z-1}{(z-1)^2 (z-2)} dz$

where C is $|z - i| = 2$ **(AU-2014)(8)**

9.a. Evaluate $\int_C \frac{z^2}{(z-1)(z+2)} dz$ where C is |z|=3 **(AU-2015) (8)**

b. Evaluate $\int_0^{2\pi} \frac{dx}{(x^2 + a^2)^2}$, a>0 using contour integration. **(AU-2015)(8)**

10.a.Evaluate $\int_0^{2\pi} \frac{\cos 3\theta}{5 - 4\cos\theta} d\theta$ using contour integration **(AU-2013)(8)**

b. Using contour integration on unit circle , evaluate $\int_0^{2\pi} \frac{d\theta}{5 + 4\cos\theta}$ **(AU-2014)(8)**

11.a. Evaluate $\int_0^{2\pi} \frac{d\theta}{13 + 5\sin\theta}$ **(AU-2014)(8)**

b. Using contour integration ,evaluate the integral $\int_0^{2\pi} \frac{\cos 2\theta}{1 - 2a\cos\theta + a^2} d\theta$ **(AU-2013)(8)**

12.a. Evaluate $\int_0^{\infty} \frac{x^2 dx}{(x^2 + a^2)(x^2 + b^2)}$, $a > 0, b > 0$ **(AU-2013)(8)**

b. Evaluate using contour integration $\int_0^{\infty} \frac{dx}{(1+x^2)^2}$ **(AU-2014)(8)-2**

∞ $x^2 dx$ π

13.a. Using contour integration prove that $\int_0 (x^2 +1)(x^2 +4) = \overline{6}$ **(AU-2013)(8)**

b. Using contour integration on unit circle , evaluate $\int_{-\infty}^{\infty} \frac{x^2\,dx}{(x^2+1)(x^2+9)}$ **(AU-2014)(8)**

14.a. Evaluate $\int_{0}^{\infty} \frac{\cos mx}{x^2+a^2}\,dx$,using contour integration. **(AU-2012)(8)**

b. Show that $\int_{-\infty}^{\infty} \frac{x^2-x+2}{(x^4+10x^2+9)}\,dx = \frac{5\pi}{2}$ **(AU-2013)(8)**

15.a.S.T. $\int_0^{\infty} \frac{dx}{(1+x^4)} = \frac{\Pi}{2\sqrt{2}}$ **(8)**

b.Evaluate $\int_{0}^{2\pi} \frac{d\theta}{1-2x\sin\theta+x^2}$ (0<x<1),using contour integration. **(8)**